K21P 4211

Reg. No. : ....

Name : ....

I Semester M.Sc. Degree (C.B.S.S. – Reg./Supple./Imp.)
Examination, October 2021
(2018 Admission Onwards)
MATHEMATICS
MAT1C03 : Real Analysis

Time : 3 Hours | Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks :

1. Let A be the set of all sequences whose elements are the digits 0 and 1. Show that A is countable.
2. If f is monotonically increasing on (a, b), show that f(x –) exists and $f(x-) \leq f(x)$ for every $x \in (a, b)$.
3. Let $f(x) = x^{10} \sin\frac{1}{x}$ if $x \neq 0$ and f(0) = 0. Is f differentiable at all points ? If so, find f′(x) for all x.
4. If f is continuous on [a, b], show that $f \in R(\alpha)$ on [a, b].
5. State and prove the integration by parts theorem.
6. Is the curve $f(t) = e^{2\pi it}$, $t \in [0, 2]$ rectifiable ? Justify. If rectifiable, find its arc length.

PART – B

Answer **any four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks :

Unit – I

7. a) Suppose X is a metric space and let $K \subset Y \subset X$. Show that K is compact relative to X if and only if K is compact relative to Y.
   b) Construct the Cantor set and show that it is perfect.
   c) If f is a continuous mapping of a metric space X into a metric space Y and if E is a connected subset of X, show that f(E) is connected.

P.T.O.

8. a) Show that every K-cell is compact.

   b) Show that a mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(V)$ is open in X for any open set V in Y.

9. a) Prove that a subset E of the real line $\mathbb{R}$ is connected if and only if it has the following property : if $x \in E$, $y \in E$ and $x < z < y$, then $z \in E$.

   b) Let f be a continuous mapping of a compact metric space X into a metric space Y. Show that f is uniformly continuous on X.

Unit – II

10. a) State and prove L'Hospital's Rule.

    b) Assume α increases monotonically and $\alpha' \in R$ on [a, b]. Let f be a bounded real function on [a, b]. Show that, $f \in R(\alpha)$ if and only if $f\alpha' \in R$ and in that case, $\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)dx$.

11. a) Suppose $f \in R(\alpha)$ on [a, b] and let $m \leq f \leq M$. A function ϕ is continuous on [m, M] and $h(x) = \phi(f(x))$ on [a, b]. Show that $h \in R(\alpha)$ on [a, b].

    b) Suppose f is bounded on [a, b]. If f has only finitely many points of discontinuity on [a, b] and if α is continuous at any point at which f is continuous, show that $f \in R(\alpha)$.

    c) Suppose $f : [a, b] \to \mathbb{R}^k$ is continuous and f is differentiable in (a, b). Show that there exists $x \in (a, b)$ such that $|f(b) - f(a)| \leq (b - a)\,|f'(x)|$.

12. a) State and prove change of variable rule in Riemann-Stieltjes integration.

    b) State and prove the generalized mean value theorem and deduce the mean value theorem.

    c) Let f and α be functions on $\left[0, \frac{\pi}{2}\right]$ defined as $f(x) = \cos x$, $\alpha(x) = \sin x$.

    Is $f \in R(\alpha)$? Justify. If $f \in R(\alpha)$ evaluate $\int_0^{\pi/2} f\,d\alpha$.

Unit – III

13. a) Let $f \in R$ on [a, b]. For $a \le x \le b$, let $F(x) = \int_a^x f(t)\,dt$. Show that F is continuous on [a, b]. Furthermore, if f is continuous at a point x_0 of [a, b], then show that F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

b) Let f be of bounded variation on [a, b]. Let $V(x) = V_f(a, x)$ if $a < x \le b$ and $V(a) = 0$. Show that every point of continuity of f is also a point of continuity of V. Prove the converse also.

c) Let $f : [a, b] \to \mathbb{R}$ satisfies $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in [a, b]$ and $K > 0$. Is f of bounded variation ? Justify.

14. a) If $f : [a, b] \to \mathbb{R}^k$ and if $f \in R(\alpha)$ for some monotonically increasing α on [a, b], show that $|f| \in R(\alpha)$ and $\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha$.

b) State and prove additive property of arc length.

c) If f is monotone increasing on [a, b], evaluate the total variation of f on [a, b].

15. a) State and prove fundamental theorem of calculus.

b) Let $f : [a, b] \to \mathbb{R}^n$ be a rectifiable path. If $x \in (a, b]$, let $s(x) = \wedge_f(a, x)$ and let $s(a) = 0$. Show that the following holds :

i) The function s is increasing and continuous on [a, b].

ii) If there is no subinterval of [a, b] on which f is constant, then s is strictly increasing on [a, b].

c) Is the function $f(x) = x \sin\frac{\pi}{x}$ if $x \ne 0$ and $f(0) = 0$ is of bounded variation on [0, 1] ? Justify.